HOW TO SOLVE A SECOND ORDER DE BY REDUCING IT TO FIRST ORDER

CASE-I

$$F(x, y', y'') = 0 \quad \text{——(1)}$$

Put $y' = z$

Diff. both sides w.r. to x

$$y'' = \frac{dz}{dx}$$

Putting these values in equation (1)

$$F\left(x, z, \frac{dz}{dx}\right) = 0 \quad \text{——(2)}$$

which is a first order DE in z and x

Solve it by variable and separable find out the value of z.

Again Put $z = y' = \frac{dy}{dx}$

Integrating both sides find out the value of y.

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NO- 3, 4, 5, 7, 8, 10, solve by reducing to first order.

4, 8

8. $xy'' + y' = 0$ —— (1)

Put $y' = z$

Diff. both sides w.r. to x

$$y'' = \frac{dz}{dx}$$

Putting these values in equation (1)

$$x\frac{dz}{dx} + z = 0$$

$$\Rightarrow x\frac{dz}{dx} = -z$$

$$\Rightarrow \frac{dz}{z} = -\frac{dx}{x}$$

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Integrating both sides,

$$\int \frac{dz}{z} = -\int \frac{dx}{x}$$

$$\Rightarrow \ln|z| = -\ln|x| + \ln c_1$$

$$\Rightarrow \ln|z| = \ln\left|\frac{c_1}{x}\right|$$

$$\Rightarrow z = \frac{c_1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{c_1}{x}$$

$$\Rightarrow dy = \frac{c_1}{x}\,dx$$

Again integrating both sides,

$$\int dy = \int \frac{c_1}{x}\,dx$$

$$\Rightarrow y = c_1 \ln|x| + c_2 .$$

CASE-II

$$F(y, y', y'') = 0 \quad \text{——— (1)}$$

Put $y' = z$

Diff. both sides w.r. to x

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy}z$$

Putting these values in equation (1)

$$F\left(y, z, \frac{dz}{dy}z\right) \quad \text{——— (2)}$$

which is a first order DE in z and y

Solve it by variable & separable find out the value of z.

Again put $z = y' = \frac{dy}{dx}$

Integrating both sides find out the value of y.

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5, 7

5. $yy'' = 2y'^2$ ——— (1)

Put $y' = z$

Diff. both sides w.r. to x

$$y'' = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{dz}{dy} z$$

Putting these values in equation (1)

$$y \frac{dz}{dy} z = 2z^2$$

$$\Rightarrow y \frac{dz}{dy} = 2z$$

$$\Rightarrow \frac{dz}{z} = 2 \frac{dy}{y}$$

Integrating both sides,

$$\int \frac{dz}{z} = 2 \int \frac{dy}{y}$$

$$\Rightarrow \ln|z| = 2\ln|y| + \ln c_1$$

$$\Rightarrow \ln|z| = \ln|y^2| + \ln c_1$$

$$\Rightarrow \ln|z| = \ln|y^2 c_1|$$

$$\Rightarrow z = y^2 c_1$$

$$\Rightarrow \frac{dy}{dx} = y^2 c_1$$

$$\Rightarrow \frac{dy}{y^2} = c_1 dx$$

Again integrating both sides,

$$\int \frac{dy}{y^2} = \int c_1 dx$$

$$\Rightarrow -\frac{1}{y} = c_1 x + c_2 .$$

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CASE-III

$$F(y', y'') = 0 \quad \text{——— (1)}$$

This equation does not contain any x or y functio which can be solved either by CASE-I or by CASE-II

3. $y'' = y'$ ——— (1)

Put $y' = z$

Diff. both sides w.r. to x

$$y'' = \frac{dz}{dx}$$

Putting these values in equation (1)

$$\frac{dz}{dx} = z$$

$$\Rightarrow \frac{dz}{z} = dx$$

Integrating both sides,

$$\int \frac{dz}{z} = \int dx$$

$$\Rightarrow \ln|z| = x + \ln c_1$$

$$\Rightarrow \ln|z| - \ln c_1 = x$$

$$\Rightarrow \ln\left|\frac{z}{c_1}\right| = x$$

$$\Rightarrow \frac{z}{c_1} = e^x$$

$$\Rightarrow z = c_1 e^x$$

$$\Rightarrow \frac{dy}{dx} = c_1 e^x$$

$$\Rightarrow dy = c_1 e^x dx$$

Again integrating both sides,

$$\int dy = \int c_1 e^x dx$$

$$\Rightarrow y = c_1 e^x + c_2 .$$

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How to find a general solution if one basis of solution is given

Let us consider the DE

$$y'' + P(x)y' + q(x)y = 0 \quad \text{——(1)}$$

Let one basis of solution y_1 is given.

Put $y_2 = uy_1$ ——(2)

Calculating the values of y_2' and y_2'' and putting it in equation (1) it reduces to

$$u = \frac{1}{y_1^2} e^{-\int P(x)\,dx} \quad \text{——(3)} \quad \text{where } u = \frac{du}{dx}$$

Putting $u = \frac{du}{dx}$ in equation (3) and integrating both sides find out u.

Putting the value of u in equation (2) find out the value of y_2.

Then the general solution is

$$y = c_1 y_1 + c_2 y_2 .$$

6, 9, 11, 12 BY E. KREYSZIG (8TH EDITION) PAGE-71

Find a general solution if one basis of solution y_1 is given.

6. $xy'' + 2y' + xy = 0 \qquad y_1 = \frac{\sin x}{x}$

Ans:- We have,

$$xy'' + 2y' + xy = 0$$

$$\Rightarrow y'' + \frac{2}{x}y' + y = 0 \quad \text{——(1)}$$

Here, $P(x) = \frac{2}{x}$, $q(x) = 1$

Put $y_2 = uy_1 = u\frac{\sin x}{x}$ ——(2)

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calculating the values of y_2' and y_2'' and putting it in equation (1) it reduces to.

$$v = \frac{1}{y_1^2} e^{-\int P(x)\,dx} \qquad \text{where} \quad v = \frac{du}{dx}$$

Now, $$\frac{du}{dx} = \frac{1}{\left(\frac{\sin x}{x}\right)^2} e^{-\int \frac{2}{x}\,dx}$$

$$= \frac{x^2}{\sin^2 x} e^{-2\ln x} = \frac{x^2}{\sin^2 x} e^{\ln x^{-2}}$$

$$= \frac{x^2}{\sin^2 x} x^{-2} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x$$

$$\Rightarrow du = \operatorname{cosec}^2 x\,dx$$

Integrating both sides,

$$\int du = \int \operatorname{cosec}^2 x\,dx$$

$$\Rightarrow u = -\cot x$$

Thus from equation (2)

$$y_2 = u y_1 = -\cot x \cdot \frac{\sin x}{x} = -\frac{\cos x}{\sin x}\,\frac{\sin x}{x}$$

$$= -\frac{\cos x}{x}$$

Hence the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$\Rightarrow y = c_1 \frac{\sin x}{x} - c_2 \frac{\cos x}{x}$$

Q. $x^2 y'' - 5x y' + 9y = 0$, $y_1 = x^3$

Ans:- We have,

$$x^2 y'' - 5x y' + 9y = 0$$

$$\Rightarrow y'' - \frac{5}{x} y' + \frac{9}{x^2} y = 0 \qquad \text{---- (1)}$$

Here, $P(x) = -\frac{5}{x}$, $q(x) = \frac{9}{x^2}$

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Put $y_2 = vy_1 = ux^3$ ——— (2)

Calculating the values of y_2' and y_2'' and putting it in equation (1) it reduces to

$$v = \frac{1}{y_1^2} e^{-\int P(x)dx} \quad \text{where} \quad v = \frac{du}{dx}$$

Now, $$\frac{du}{dx} = \frac{1}{(x^3)^2} e^{-\int -\frac{5}{x} dx} = \frac{1}{x^6} e^{\int \frac{5}{x} dx}$$

$$= \frac{1}{x^6} e^{5 \ln|x|} = \frac{1}{x^6} e^{\ln|x^5|}$$

$$= \frac{1}{x^6} x^5 = \frac{1}{x}$$

$$\Rightarrow du = \frac{1}{x} dx$$

Integrating both sides,

$$\int du = \int \frac{dx}{x}$$

$$\Rightarrow u = \ln|x|$$

Thus from equation (2)

$$y_2 = uy_1 = (\ln|x|) x^3 = x^3 \ln|x|$$

Hence, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$\Rightarrow y = c_1 x^3 + c_2 x^3 \ln|x|$$

$$\Rightarrow y = (c_1 + c_2 \ln|x|) x^3.$$

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